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SOME USES OF GREEN'S THEOREM IN SOLVING THE NAVIER–STOKES EQUATIONS

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SUMMARY

This paper gives a review of methods where Green's theorem may be employed in solving numerically the Navier–Stokes equations for incompressible fluid motion. They are based on the concept of using the theorem to transform local boundary conditions given on the boundary of a closed region in the solution domain into global, or integral, conditions taken over it. Two formulations of the Navier–Stokes equations are considered: that in terms of the streamfunction and vorticity for two-dimensional motion and that in terms of the primitive variables of the velocity components and the pressure. In the first formulation overspecification of conditions for the streamfunction is utilized to obtain conditions of integral type for the vorticity and in the second formulation integral conditions for the pressure are found. Some illustrations of the principle of the method are given in one space dimension, including some derived from two-dimensional flows using the series truncation method. In particular, an illustration is given of the calculation of surface vorticity for two-dimensional flow normal to a flat plate. An account is also given of the implementation of these methods for general two-dimensional flows in both of the mentioned formulations and a numerical illustration is given.

KEY WORDS Navier–Stokes Integral conditions Vorticity Pressure

INTRODUCTION AND ONE-DIMENSIONAL ILLUSTRATIONS

Green's theorem, or the identities associated with it, is fundamental in the treatment of vector fields. By means of the theorem, line integrals round closed curves in a plane can be expressed as double integrals over the domain bounded by them, or *vice versa.* There are numerous methods of employing these results in solving plane potential, biharmonic and Navier–Stokes problems. One example is the boundary integral method, to which many references may be found. An account of some recent applications of this method has been given by Ingham and Kelmanson.[1] The use of Green's theorem and associated integral theorems is also basic in the finite element method, which is now an important technique in the solution of the above mentioned classes of problems. These applications are well known and will not be considered in the present paper.

The basic application to be considered here is the use of Green's identity to transform locally specified boundary conditions in Navier–Stokes and biharmonic problems into alternative conditions of a global nature. For example, when the vorticity–streamfunction formulation is used

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in two-dimensional problems of the Navier–Stokes type, the boundary conditions for the streamfunction on solid boundaries in the fluid are overspecified while none is given for the vorticity. However, the overspecified local conditions can be utilized to obtain global conditions for the vorticity. We can illustrate this very simply with a one-dimensional model. Let the functions $\psi(x)$ and $\zeta(x)$ satisfy the differential equations

$$\zeta'' + f(x)\zeta' = g(x), \tag{1}$$

$$\psi'' = \zeta, \tag{2}$$

with boundary conditions

$$\psi = \psi' = 0 \quad \text{when } x = 0, l. \tag{3}$$

The prime denotes differentiation with respect to x; the functions $f(x)$ and $g(x)$ could either be given or depend upon ψ, in which case the problem is non-linear. Such a case could be a simple model of the Navier–Stokes equations in one space dimension.

The fundamental solutions of $\psi'' = 0$ are simply $\psi = 1$ and $\psi = x$, and if we multiply (2) by each of these in turn and integrate with respect to x from $x = 0$ to $x = l$, we obtain the two conditions

$$\int_0^l \zeta \mathrm{d}x = 0; \qquad \int_0^l x\zeta \mathrm{d}x = 0. \tag{4a, b}$$

Provided that the conditions (4a, b) have been satisfied, equation (2) may be integrated subject to any two of the conditions (3). This can be illustrated by integrating (2) twice in succession with respect to x, which gives the equations

$$\psi'(x) = \psi'(0) + \int_0^x \zeta(\xi)\mathrm{d}\xi \tag{4c}$$

and

$$\psi(x) = \psi(0) + x\psi'(0) + \int_0^x (x - \xi)\zeta(\xi)\mathrm{d}\xi \tag{4d}$$

respectively. In solving (2) numerically it is common to employ Dirichlet-type conditions $\psi(0) = \psi(l) = 0$. If such a solution has been found and (4a, b) have been satisfied, it follows by putting $x = l$ in (4d) that the condition $\psi'(0) = 0$ must be satisfied. We then find that $\psi'(l) = 0$ is satisfied by putting $x = l$ in (4c). Use of the conditions $\psi(0) = \psi'(l) = 0$ with (2) is similarly justified.

Equally, however, we could convert the numerical solution procedure for (2) into an initial value problem by choosing the conditions $\psi(0) = \psi'(0) = 0$. As long as an accurate enough step-by-step method is used and (4a, b) are satisfied, it again follows from (4c) that $\psi'(l) = 0$ and from (4d) that $\psi(l) = 0$. In fact we may use this latter boundary condition in order to give a very useful check on the numerical procedure, i.e. the step-by-step method must be such that this condition does come out to be satisfied. An example illustrating this situation is given later.

The creation of an initial value problem was considered by Dennis and Chang[2] for a slightly more complicated equation than (2) in which integral conditions were used in the solution. The equation arises in the solution of Poisson's equation by Fourier analysis and, in that case, the created initial value problem was unstable and required special treatment. Specialized integration methods were necessary, using a factorization procedure into two first-order equations, with a forward integration process for one and a backward integration process for the other. Nevertheless, the concept of treating an essentially boundary value type of problem by initial value techniques when the boundary conditions are overspecified is of some interest. Finally, in the present case it may be noted that the function and its derivative need not be zero in (3) but could be any constants, with the zeros in (4a, b) replaced by $\psi'(l)-\psi'(0)$, $l\psi'(l)-\psi(l)+\psi(0)$. This is easily verified from (4c, d).

The use of (4a, b) in a finite difference approach to the solution of (1) and (2) is reasonably obvious. For example, the central difference, h^2-accurate approximations at a grid point $x=x_0$ are

$$\{1-\tfrac{1}{2}hf(x_0)\}\zeta(x_0-h)-2\zeta(x_0)+\{1+\tfrac{1}{2}hf(x_0)\}\zeta(x_0+h)=h^2g(x_0), \tag{5}$$

$$\psi(x_0-h)-2\psi(x_0)+\psi(x_0+h)=h^2\zeta(x_0), \tag{6}$$

where h is the grid size. The customary approach is to use the conditions $\psi(0)=\psi(l)=0$ in conjunction with the set of equations (6) and to create boundary conditions of Dirichlet type for the set of equations (5) by utilizing the central difference approximations to $\psi'(0)=0$, $\psi'(l)=0$ in conjunction with (6) locally at each end to give

$$\zeta(0)=2\psi(h)/h^2, \qquad \zeta(l)=2\psi(l-h)/h^2. \tag{7}$$

The conditions (7) are only h-accurate approximations, but h^2-accurate formulae can be obtained following the method of Woods.[3] These are found to be

$$\zeta(0)=3\psi(h)/h^2-\tfrac{1}{2}\zeta(h), \qquad \zeta(l)=3\psi(l-h)/h^2-\tfrac{1}{2}\zeta(l-h). \tag{8}$$

In either case of (7) or (8) the numerical solutions for ζ and ψ are coupled not only through the sets of equations (5) and (6) but also through the local calculation of boundary conditons. On the contrary, if the conditions (4a, b) are employed, the coupling of ψ and ζ is eliminated when the problem is linear. From a practical point of view, each of the conditions (4a, b) can be expressed as a quadrature formula in the form

$$\sum_{n=0}^{N} c_n\zeta(nh)=0, \tag{9}$$

where $Nh=l$; i.e. there are $N+1$ grid points in all, including boundary points. The coefficients c_n will depend upon the quadrature formula used and also upon which (4a, b) is being represented. Then, with this understanding, there are two equations of type (9) to supplement the set (5) and this completes the formulation. These two conditions replace either the two conditions given in (7) or in (8). A possible disadvantage of formulae of type (9) is that they involve, in general, every component of the vector to be determined. However, one may use a tridiagonal reduction of (5) to express $\zeta(nh)$ $(n=1,2,\ldots,N-1)$ in terms of $\zeta(0)$ and $\zeta(l)$. Then substitution in the two equations of type (9) determines $\zeta(0)$ and $\zeta(l)$ by elimination. Methods of this type have been considered by Dennis and Quartapelle.[4]

We can use quadrature formulae of any order of accuracy to obtain formulae of type (9), and thus if finite difference methods are used, we can utilize higher-order-accurate approximations to (1) and (2) to obtain uniformly accurate approximations. For example, an h^4-accurate approximation to (1) has been given by Dennis[5] in the form

$$\begin{aligned} &c(x_0-h)\zeta(x_0-h)-c(x_0)\zeta(x_0)+c(x_0+h)\zeta(x_0+h)\\ &\quad=h^2\{(1-\tfrac{1}{2}hf_0)g(x_0-h)+10g(x_0)+(1+\tfrac{1}{2}hf_0)g(x_0+h)\}/12, \end{aligned} \tag{10}$$

where

$$\begin{aligned} c(x_0\pm h)&=1\pm\tfrac{1}{2}hf_0+h^2(f_0^2+2f'_0)/12\pm h^3(f_0f'_0+f''_0)/24,\\ c(x_0)&=2+h^2(f_0^2+2f'_0)/6 \end{aligned} \tag{11}$$

and the subscript zero denotes the value at $x=x_0$ of quantities on the right-hand sides of equations (11). The standard Numerov h^4-accurate approximation to (2) is

$$\psi(x_0-h)-2\psi(x_0)+\psi(x_0+h)=h^2\{\zeta(x_0-h)+10\zeta(x_0)+\zeta(x_0+h)\}/12, \tag{12}$$

and with the use of Simpson's rule for (9) we can get a uniform h^4-accurate approximation without any special treatment of the end points. An example of a trial calculation will be given later.

APPLICATIONS USING THE METHOD OF SERIES TRUNCATION

The first applications of integral conditions of the kind discussed in the present paper were in fact made to problems involving the two-dimensional steady flow of an incompressible fluid past a circular cylinder and past a flat plate of finite length aligned with the direction of the stream.[6–9] In these applications the method of series truncation was utilized to reduce the governing partial differential equations to sets of ordinary differential equations. The formulation using the streamfunction and vorticity was employed and integral conditions involving the vorticity were used in solving sets of one-dimensional equations governing the components in Fourier series expansions.

The equations governing the dimensionless scalar vorticity ζ and streamfunction ψ for these problems can be written

$$\partial^2\zeta/\partial\xi^2+\partial^2\zeta/\partial\theta^2=\tfrac{1}{2}Re\,J(\zeta,\psi), \tag{13}$$

$$\partial^2\psi/\partial\xi^2+\partial^2\psi/\partial\theta^2=-M^2\zeta, \tag{14}$$

where $J(\zeta,\psi)$ is the usual Jacobian in terms of the co-ordinates (ξ,θ), Re is a Reynolds number and M is a metric associated with the co-ordinate system. For a cylinder (ξ,θ) are modified polar co-ordinates with $\xi=\ln(r/a)$, where a is the radius of the cylinder. The metric is e^{ξ} and the Reynolds number is $Re=2aU/\nu$, where U is the uniform stream parallel to the direction $\theta=0$ and ν is the coefficient of kinematic viscosity. For an aligned flat plate the Cartesian co-ordinates (x,y) are related to (ξ,θ) by $x=a\cosh\xi\cos\theta$, $y=a\sinh\xi\sin\theta$, where $2a$ is the length of the plate. The metric is given by $M^2=\tfrac{1}{2}(\cosh 2\xi-\cos 2\theta)$ and again $Re=2aU/\nu$. In both cases the flow is symmetrical about the axis of x. The boundary conditions are

$$\psi=\partial\psi/\partial\xi=0 \quad \text{when } \xi=0, \tag{15}$$

$$e^{-\xi}\partial\psi/\partial\xi\to k\sin\theta, \quad e^{-\xi}\partial\psi/\partial\theta\to k\cos\theta \quad \text{as } \xi\to\infty, \tag{16}$$

$$\psi=\zeta=0 \quad \text{when } \theta=0,\pi. \tag{17}$$

The constant k depends upon the case under consideration; we have $k=1$ for a circular cylinder and $k=\tfrac{1}{2}$ for a flat plate. In both cases the conditions (16) express the free stream conditions at large distances. The domain of the symmetrical flow is $\xi\geqslant 0$, $0\leqslant\theta\leqslant\pi$.

The series assumed in References 6–9 for the streamfunction is

$$\psi=\sum_{n=1}^{\infty}f_n(\xi)\sin n\theta, \tag{18}$$

which may then be differentiated twice with respect to θ because of the conditions (17). Equation (14) then gives

$$f_n''-n^2f_n=r_n(\xi) \quad (n=1,2,\ldots), \tag{19}$$

where

$$r_n(\xi) = -(2/\pi)\int_0^\pi M^2\zeta\sin n\theta\,\mathrm{d}\theta. \tag{20}$$

The boundary conditions associated with (19) are obtained from (15) and (16) as

$$f_n = f'_n = 0 \quad \text{when } \xi = 0, \tag{21}$$

$$\mathrm{e}^{-\xi}f_n(\xi)\to k\delta_{n,1}, \quad \mathrm{e}^{-\xi}f'_n(\xi)\to k\delta_{n,1} \quad \text{as } \xi\to\infty, \tag{22}$$

where $\delta_{n,1}$ is the Kronecker delta. Thus the conditions for the streamfunction are overspecified at $\xi=0$ and no condition is known there for ζ. On the other hand, the free stream conditions (22) imply that $\zeta\to 0$ as $\xi\to\infty$.

The fundamental solutions associated with the operator on the left-hand side of equations (19) are $\mathrm{e}^{\pm n\xi}$. The solution with exponent $+n\xi$ is not useful in the infinite domain. If we multiply (19) by the other solution and integrate from $\xi=0$ to $\xi=\infty$, we obtain, using (21),

$$\int_0^\infty \mathrm{e}^{-n\xi}r_n(\xi)\mathrm{d}\xi = 2k\delta_{n,1} \quad (n=1,2,\ldots). \tag{23}$$

This set of integral conditions may be used by means of a suitable quadrature formula to relate $r_n(0)$ to internal values of $r_n(\xi)$ for each value of n. It was used in References 6–9 to calculate values of $r_n(0)$ from the computed vorticity distribution in the solution domain by evaluating (20) for each $\xi\neq 0$ and then using the appropriate quadrature formula to approximate (23). After computation of these values, the surface vorticity is calculated from inversion of (20) in the form

$$\zeta(\xi,\theta) = -(1/M^2)\sum_{n=1}^{\infty} r_n(\xi)\sin n\theta, \tag{24}$$

which is evaluated at $\xi=0$. Of course the infinite limits in both (23) and (24) must be replaced by finite values in order to carry out the calculations and these become parameters of the problem. Each must be chosen large enough to give an adequate approximation to the problem and each must be varied to check the adequacy of the approximation.

It may be noted that only one set of integral conditions is necessary in problems of this kind since it is necessary to determine the vorticity only on $\xi=0$. The vorticity must vanish as $\xi\to\infty$ and an approximation to it at some large enough value of ξ must be made; this is discussed in References 6–9. Once the conditions (23) are satisfied, the set of equations (19) can in theory be solved as initial value problems subject to the conditions (21). However, the solution procedures are highly unstable and were treated in References 8 and 9 by factorizing the operator in (19) and integrating each factorized equation in opposite directions using step-by-step techniques. A detailed discussion is given in Reference 2; this problem has already been mentioned in the previous section. The common feature of all of References 2 and 6–9 is that the set of equations (19) is solved to determine the streamfunction and the integral conditions (23) are used to determine the boundary vorticity. Specialized techniques are necessary to determine $r_n(\xi)$ from (20), especially if n is large, and these are described in Reference 2.

One particular advantage of the technique described in the case of flow past a flat plate is that it deals effectively with the leading and trailing edge singularities. The vorticity becomes infinite at these points, but $M^2\zeta$ exists there. Moreover, ζ is only infinite when $\xi=0$, so there is no problem in evaluating the integral in (20) when $\xi\neq 0$. Thus the integral conditions can still be employed to calculate $r_n(0)$. The surface vorticity is then calculated from (24) and the singularities in ζ at $\xi=0$, $\theta=0$ and $\xi=0$, $\theta=\pi$ enter only on division by M^2 in (24). A very similar but rather more difficult case is presented by flow normal to a flat plate. Here we can express the Cartesian co-ordinates (x, y) as

$$x = a \sinh\xi \cos\theta, \qquad y = a \cosh\xi \sin\theta \tag{25}$$

and the metric in (14) is given by $M^2 = \frac{1}{2}(\cosh 2\xi + \cos 2\theta)$. The plate now occupies the position $x=0$, $-a \leqslant y \leqslant a$, whereas for the aligned case it occupied the position $-a \leqslant x \leqslant a$, $y=0$. In both cases the uniform stream U at large distances is parallel to the positive direction of the axis of x, the Reynolds number in (13) is $Re = 2aU/\nu$ and the region $y \geqslant 0$ of flow is the region $\xi \geqslant 0$, $0 \leqslant \theta \leqslant \pi$ of the transformed plane.

With this formulation the boundary conditions (15)–(17) are exactly the same for this case of flow normal to a flat plate provided $k = \frac{1}{2}$. In fact the substitution (18) may equally be made in this case and all the ensuing equations (19)–(24) are the same, with the understanding of the revised definition of the metric M. The singularity in ζ now appears at the point $\xi = 0$, $\theta = \pi/2$ of the region of computation, which is taken as the region $\xi \geqslant 0, 0 \geqslant \theta \geqslant \pi$. This causes more problems than in the case of the aligned flat plate. We shall not go into the reasons in detail except to say that it is due to the more exposed position of the point $\xi = 0$, $\theta = \pi/2$ in the domain of computation in this case. In other words, the singularity exerts more influence on the computation of the vorticity. In the final section of the present paper we shall give a brief comparison of results for the surface vorticity for flow normal to a flat plate at $Re = 70$. One result has been obtained using standard finite difference procedures throughout; the other has been obtained using integral conditions. Some similar comparisons are given in the case of flow past a circular cylinder.

Numerous other applications of integral conditions in a one-dimensional form have been made following the use of series truncation methods. For example, Dennis and Walker[10] made an early study of low-Reynolds-number flow past a sphere by these techniques and were able to obtain extremely accurate results. Dennis and Singh[11] used integral conditions in computing flow between two rotating spheres. Dennis *et al.*[12] used similar conditions in computing flow external to a rotating sphere. Dennis and Ng[13] computed steady flow through a curved tube and found dual solutions. Integral conditions are also equally applicable in problems of unsteady flow. Dennis and Walker[14, 15] considered the problem of finding the unsteady flow past an impulsively started sphere using these methods. Collins and Dennis[16,17] have shown that exactly the same set of conditions (23) are applicable to unsteady flow past an impulsively started circular cylinder. They were used in Reference 16 to determine the initial flow in the boundary layer mainly by analytical methods and in Reference 17 the investigation was continued mainly by numerical methods. In a more recent investigation, Badr and Dennis[18] have shown that the satisfaction of integral conditions is closely connected with the maintenance of the correct circulation round contours at large distances surrounding a suddenly started rotating and translating circular cylinder and that it is necessary for one integral condition in particular to be satisfied for the pressure round the surface of the cylinder to be single-valued. Integral conditions have also been used recently by Anwar and Dennis[19] in an investigation of flow generated by moving walls using the series truncation method.

APPLICATIONS TO TWO-DIMENSIONAL EQUATIONS

The previous sections have dealt essentially with integral conditions in one space dimension. The integral conditions can be generalized to two-dimensional equations in situations involving arbitrary domains and any type of boundary conditions. In such cases it is not possible to use series methods to reduce the problems to one-dimensional analogues. The extension to these situations is made possible by application of the Green's identity appropriate to the elliptic partial differential operator which is obtained when a basically fourth-order equation is factorized into two second-order equations. We shall examine two possible formulations of the Navier–Stokes equations for incompressible fluids in two dimensions, firstly that in terms of the streamfunction and vorticity and then that in terms of the velocity–pressure equations. It will be seen that the cases considered in previous sections may be looked upon as nothing more than special cases of this more general approach.

Vorticity–streamfunction formulation

The Navier–Stokes equations for the unsteady plane motion of an incompressible viscous fluid can be written in terms of the vorticity ζ and streamfunction ψ in the form

$$\partial\zeta/\partial t + J(\zeta, \psi) = \nu\nabla^2\zeta, \tag{26}$$

$$\nabla^2\psi = -\zeta, \tag{27}$$

where t is the time. The boundary conditions most frequently associated with (26), (27) come from the specification of the velocity vector $\mathbf{u}$ on the boundary S of a flow domain V. Since the velocity vector has Cartesian components $(\partial\psi/\partial y, -\partial\psi/\partial x)$, we can determine from the specification of $\mathbf{u}$ the two conditions

$$\psi|_S = a(s,t), \qquad \partial\psi/\partial n|_S = b(s,t), \tag{28}$$

where $\mathbf{n}$ is the outward normal to S and s is measured along it. The function $a(s, t)$ is obtained by integrating in the direction of s the component of velocity along the outward normal to S and $b(s, t)$ is the negative of the tangential velocity component. The function $a(s, t)$ is arbitrary to the extent of an additive constant, but is single-valued since the boundary velocity vector is assumed to satisfy a global incompressibility condition over the domain V.

As we have already mentioned, the boundary conditions (28) pose a problem in the solution of (26), (27) in that two conditions are specified for ψ and none for ζ. In previous sections it has been demonstrated how this overspecification can be overcome by the use of integral conditions of one-dimensional character. In the two-dimensional case[20,21] we can derive integral conditions by using Green's identity for the Laplacian operator, namely

$$\int(\phi\nabla^2\psi - \psi\nabla^2\phi)\mathrm{d}V = \oint(\phi\,\partial\psi/\partial n - \psi\,\partial\phi/\partial n)\mathrm{d}s. \tag{29}$$

If we substitute $\nabla^2\psi = -\zeta$ from (27) and choose $\phi = \eta$, where η is a harmonic function, we readily find by applying the identity to the given domain of flow that

$$\int\zeta(\mathbf{r}, t)\eta(\mathbf{r})\mathrm{d}V = \oint[a(s, t)(\partial\eta/\partial n)_{\mathbf{r}=\mathbf{r}_S} - b(s, t)\eta(\mathbf{r}_S)]\mathrm{d}s, \tag{30}$$

where $\mathbf{r}$ is the position vector of an internal point of V and $\mathbf{r}_S$ is a point lying on S. The number of conditions (30) to be satisfied will depend upon the number of harmonic functions utilized and this in turn depends upon the manner in which the problem is discretized spatially.

For example, if we consider the problem of flow past a circular cylinder or a flat plate of the previous section, we can choose $\eta = \mathrm{e}^{-n\xi}\sin n\theta$ $(n = 1, 2, \ldots)$. The boundary S of the flow domain is taken as the contour of the cylinder or plate itself together with a contour $\xi = \xi_\infty$ at large distances from it. The contribution to the line integral on the right-hand side of (30) due to the cylinder or plate itself is zero by virtue of the conditions (15). The conditions (16) can be combined as

$$\psi \sim k\mathrm{e}^{\xi}\sin\theta \quad \text{as} \quad \xi \to \infty, \tag{31}$$

so that if ξ_∞ is large enough, the condition (30) reduces to

$$\int_0^{2\pi}\int_0^{\xi_\infty} M^2\zeta e^{-n\xi}\sin n\theta d\xi d\theta \sim -2k\int_0^{2\pi} e^{(1-n)\xi}\sin\theta \sin n\theta d\theta. \qquad (32)$$

If we substitute (24), evaluate the integrals in (32) and let $\xi_\infty \to \infty$, we get the set of conditions (23). As we have already explained, the number of terms employed in (24) to approximate the infinite sum is a parameter of the series truncation method and must generally be made large enough to give a good numerical approximation to the solution.

In the general two-dimensional case when series reductions such as those of the previous section are not employed, the conditions (30) provide a set of two-dimensional integral conditions by which to determine the boundary vorticity. The number of conditions to be satisfied, corresponding to the number of linearly independent harmonic fields $\eta(\mathbf{r})$, implies the use of as many conditions as the number of boundary points at which the vorticity must be determined. The use of a given finite number of boundary points and corresponding conditions gives an approximation to the solution and the underlying principle is that the degree of approximation is improved as more boundary points and more harmonic fields are employed. Thus equation (26) is always supplemented with the correct number of conditions to give a well determined problem.

An important aspect of the conditions (30) is their integral and therefore non-local character in which the vorticity distribution on the boundary is characterized by its distribution in the interior of the domain. In other words, the interior vorticity distribution imposes a constraint on the boundary vorticity. This property emerges also from a geometrical interpretation of the vorticity integral conditions. If for simplicity we consider the homogeneous case $a=b=0$, the integral conditions impose the orthogonality (in the abstract space sense) of the vorticity field with respect to the linear manifold of the harmonic functions in the domain V. This means that in order that the vorticity field shall be compatible with the velocity prescribed on S, an operation of orthogonal projection must be performed and such an operation has certainly a non-local character. We note that the conditions (30) provide a mathematical description of the generation of boundary vorticity in viscous flows which is essentially non-local. It is also worth mentioning that in the case of the trivial harmonic function $\eta(\mathbf{r})\equiv 1$, the condition (30) gives the Stokes theorem for plane flows:

$$\int \zeta(\mathbf{r}, t)dV = -\oint b(s, t)ds. \qquad (33)$$

If we turn now to the question of finding a discretized version of the problem, there are several ways of satisfying the vorticity integral conditions. The first and most obvious method is to use the condition (30) directly to give a linear equation relating all point values of the unknown ζ, with coefficients equal to the point values of the harmonic functions $\eta(\mathbf{r})$. The set of such linear equations corresponding to the manifold of the discrete set of harmonic functions then closes the system of algebraic equations resulting from the discretization of the vorticity transport equation (26). Unfortunately, this full system of equations has a rather cumbersome profile since the equations expressing the integral conditions have almost all coefficients different from zero. This method has therefore not been used in two dimensions and has been considered only for one-dimensional representations such as those of the spectral type given by Dennis and Quartapelle.[22]

The difficulty of this cumbersome profile can be eliminated by a method which makes use of the superposition principle. As an illustration we shall consider the case of a time-discretized version of the vorticity–streamfunction equations (26) and (27) subject to integral conditions. The equations determining the unknown $\zeta=\zeta^{n+1}$, $\psi=\psi^{n+1}$ at the new time level $t^{n+1}=t^n+\Delta t$ are

$$(-\nabla^2+\gamma)\zeta=f, \qquad \int\zeta\eta \mathrm{d}V=\oint(a\partial\eta/\partial n-b\eta)\mathrm{d}s, \tag{34}$$

$$-\nabla^2\psi=\zeta, \qquad \psi|_S=a \quad \text{or} \quad \partial\psi/\partial n|_S=b, \tag{35}$$

where

$$\gamma=1/(\nu\Delta t), \qquad f=\gamma\zeta^n-\nu^{-1}J(\zeta^n,\psi^n),$$
$$a=a(s)=a(s,t^{n+1}), \qquad b=b(s)=b(s,t^{n+1}).$$

The unknown $\zeta(\mathbf{r})$ is decomposed following the procedure of Quartapelle[20] in the form

$$\zeta(\mathbf{r})=\zeta_0(\mathbf{r})+\oint\zeta'(\mathbf{r},s')\lambda(s')\mathrm{d}s'. \tag{36}$$

Here the fields $\zeta_0(\mathbf{r})$ and $\zeta'(\mathbf{r},s')$, where s' is contained in S, are the solutions of the problems

$$(-\nabla^2+\gamma)\zeta_0=f, \qquad \zeta_0|_S=\text{arbitrary} \tag{37}$$

and

$$(-\nabla^2+\gamma)\zeta'=0, \qquad \zeta'|_S=\delta(s-s') \tag{38}$$

respectively and δ is the Dirac delta function. The unknown $\lambda(s)$, where s is contained in S, is determined by solving the linear problem obtained by requiring $\zeta(\mathbf{r})$ to satisfy the integral condition given in equation (34). This gives

$$\oint A(s,s')\lambda(s')\mathrm{d}s'=\beta(s), \tag{39}$$

where

$$A(s,s')=\int\zeta'\eta\mathrm{d}V, \tag{40}$$

$$\beta(s)=-\int\zeta_0\eta\mathrm{d}V+\oint(a\partial\eta/\partial n-b\eta)\mathrm{d}s. \tag{41}$$

Such a computational scheme is therefore based on the solution of only Dirichlet problems for the operators $-\nabla^2+\gamma$ and ∇^2, together with an additional linear problem (39) to determine the unknown λ on the boundary. In the spatially discretized case it can be shown that the matrix corresponding to the linear operator $A(s,s')$ is symmetric. Equations (40) and (41) have been used in this form in the earlier implementations of the vorticity integral conditions.[20,21] There is, however, the disadvantage that the harmonic functions η must be stored in the computer memory and that the evaluation of the integrals in (40) and (41) is time-consuming.

Fortunately, both inconveniences are eliminated by the method proposed by Glowinski and Pironneau[23] for the solution of the biharmonic problem. Their method doubles the number of elliptic problems to be solved and in the present case of the time-discretized vorticity–streamfunction equations requires that the problems

$$-\nabla^2\psi_0=\zeta_0, \qquad \psi_0|_S=a, \tag{42}$$

$$-\nabla^2\psi'=\zeta', \qquad \psi'|_S=0 \tag{43}$$

be solved after the problems (37) and (38). Then, by introducing for each s contained in S the simple function

$$w = \text{arbitrary in } V, \qquad w|_S = \delta(s - \bar{s}), \tag{44}$$

it is possible to characterize the quantities A and β in the equivalent form

$$A(s, s') = \int (\zeta' w - \nabla\psi' \cdot \nabla w) \mathrm{d}V, \tag{45}$$

$$\beta(s) = -\int (\zeta_0 w - \nabla\psi_0 \cdot \nabla w) \mathrm{d}V - \oint b w \mathrm{d}s. \tag{46}$$

The arbitrariness of the functions w at all internal points of V is exploited by choosing $w = 0$ inside V so that the integration domain in the relations (45) and (46) is reduced to a narrow strip along the boundary.[23] A finite element method which enforces the vorticity integral conditions by means of the Glowinski–Pironneau method has been described by Quartapelle and Napolitano,[24] while a finite difference implementation of the method for axisymmetric flows has been considered by Dennis and Quartapelle.[25]

A third method for a direct, i.e. non-iterative, determination of the vorticity on the boundary is the so-called influence matrix method.[26] It is very similar to the method just described and leads to a linear problem for a surface unknown which is analogous to that expressed by equation (39). A fourth method for the solution of the Navier–Stokes equations in non-primitive variables consists of solving the vorticity and streamfunction equations coupled together so that it is possible to impose both the Dirichlet and the Neumann conditions simultaneously.[27] A quite similar approach in principle can be used also in conjunction with a finite element type of spatial discretization which considers the Neumann condition as a natural condition for the streamfunction equation and a Dirichlet condition as an essential condition for the vorticity equation.[28] The vorticity field computed by this method satisfies the integral conditions as a result of the complicated influence of the imposition of both conditions for ψ through the coupling of the equations.

The coupled equation approach was pioneered by Davis[27] using finite differences and the ADI technique. It must be noted that in this case the intrinsically two-dimensional character of the vorticity conditioning is achieved very ingeniously only at the end and as a consequence of the iterative procedure. Another method which still achieves iteratively the satisfaction of integral conditions for the vorticity is the so-called decoupled equation approach for the biharmonic equation.[29] Here the Dirichlet problems for the second-order elliptic equations are solved iteratively with the vorticity boundary values approximated in a convenient way. By choosing the relaxation parameter for the evaluation of the boundary vorticity in a proper range, the iteration scheme is made to converge to the solution of the fourth-order problem. Therefore in this approach the effect of non-locality associated with the integral conditions is modelled by the iterative and sequential solution of the elliptic equations. It may be noted that this interpretation of the decoupled equation method provides an explanation of the sometimes observed superiority of boundary vorticity formulae of low accuracy with respect to higher-accuracy approximations. In fact the effectiveness of a boundary vorticity formula must be measured not necessarily on the basis of its accuracy, which is local, but rather on its capability of converging rapidly by a relaxation process to the satisfaction of non-local conditions.

Velocity–pressure formulation

The concept of integral conditions applies also to the formulation of the Navier–Stokes equations for the motion of incompressible fluids when expressed in terms of the velocity components and the pressure as dependent variables. Again as an example we consider the time-discretized equations for the unknowns $\mathbf{u}=\mathbf{u}^{n+1}$, $p=p^{n+1}$ at the new time level t^{n+1}, where $\mathbf{u}$ is the velocity vector. These can be expressed as

$$(-\nabla^2+\gamma)\mathbf{u}+\nabla p=\gamma\mathbf{u}^n+\mathbf{f}, \tag{47}$$

$$-\nabla^2 p=-\nabla\cdot\mathbf{f}, \tag{48}$$

where

$$\gamma=1/(\nu\Delta t), \qquad \mathbf{f}=-\nu^{-1}(\mathbf{u}^n\cdot\nabla)\mathbf{u}^n.$$

The Poisson equation (48) for the pressure has been obtained by taking the divergence of the momentum equation and using the equation $\nabla\cdot\mathbf{u}=0$. Kleiser and Schumann[30] have shown that in order to ensure an exact satisfaction of the incompressibility condition when using the Poisson equation (48), the equation $\nabla\cdot\mathbf{u}=0$ must be retained on the boundary. Then the boundary conditions for equations (47) and (48) are

$$\mathbf{u}|_S=\mathbf{b}, \qquad \nabla\cdot\mathbf{u}|_S=0, \tag{49}$$

where $\mathbf{b}=\mathbf{b}(s, t^{n+1})$. The situation is very similar to the case when non-primitive variables are used. There are too many conditions for one variable, the velocity, and none for the other, the pressure. It is possible in much the same manner to obtain the missing conditions for the pressure.[31]

We consider the vector version of the Green's identity for the Helmholtz operator $-\nabla^2+\gamma=-\nabla_\gamma^2$, namely

$$\int(\mathbf{v}\cdot\nabla_\gamma^2\mathbf{u}-\mathbf{u}\cdot\nabla_\gamma^2\mathbf{v})\mathrm{d}V=\oint[(\mathbf{n}\cdot\mathbf{v})(\nabla\cdot\mathbf{u})-(\mathbf{n}\cdot\mathbf{u})(\nabla\cdot\mathbf{v})+(\mathbf{n}\times\mathbf{v})\cdot(\nabla\times\mathbf{u})-(\mathbf{n}\times\mathbf{u})\cdot(\nabla\times\mathbf{v})]\mathrm{d}s. \tag{50}$$

By introducing the vector fields $\chi_\gamma(\mathbf{r})$, which are solutions of the Helmholtz problem

$$(-\nabla^2+\gamma)\chi_\gamma=0, \quad (\mathbf{n}\cdot\chi_\gamma)|_S\neq 0, \quad (\mathbf{n}\times\chi_\gamma)|_S=0 \tag{51}$$

it is possible to obtain an integral condition for the pressure gradient in the form

$$\int\nabla p\cdot\chi_\gamma\,\mathrm{d}V=\int(\gamma\mathbf{u}^n+\mathbf{f})\cdot\chi_\gamma\mathrm{d}V-\oint[(\mathbf{n}\cdot\mathbf{b})(\nabla\cdot\chi_\gamma)+(\mathbf{n}\times\mathbf{b})\cdot(\nabla\times\chi_\gamma)]\mathrm{d}s. \tag{52}$$

Since the number of linearly independent fields χ_γ taken as solutions of (51) is equal to the number of boundary points, the integral conditions (52) combined with the Poisson equation (48) for the pressure provide a well determined problem. The pressure is of course arbitrary to the extent of an added constant for incompressible flows.

In order to obtain a solution of (48) subject to (52), we separate p into its harmonic and non-harmonic components by writing

$$p(\mathbf{r})=p_0(\mathbf{r})+\oint p'(\mathbf{r}, s')\lambda(s')\mathrm{d}s', \tag{53}$$

where $p_0(\mathbf{r})$ and $p'(\mathbf{r}, s')$ are the solutions of

$$-\nabla^2 p_0=-\nabla\cdot\mathbf{f}, \qquad p_0|_S=\text{arbitrary}, \tag{54}$$

$$-\nabla^2 p'=0, \qquad p'|_S=\delta(s-s'). \tag{55}$$

The unknown $\lambda(s)$ is determined by solving the linear problem obtained by requiring it to satisfy the integral condition (52), which gives

$$\oint A(s, s')\lambda(s')\mathrm{d}s' = \beta(s), \tag{56}$$

where

$$A(s, s') = \int \nabla p' \cdot \boldsymbol{\chi}_\gamma \mathrm{d}V, \tag{57}$$

$$\beta(s) = -\int \nabla p_0 \cdot \boldsymbol{\chi}_\gamma \mathrm{d}V + \int (\gamma \mathbf{u}^n + \mathbf{f}) \cdot \boldsymbol{\chi}_\gamma \mathrm{d}V - \oint [(\mathbf{n} \cdot \mathbf{b})(\nabla \cdot \boldsymbol{\chi}_\gamma) + (\mathbf{n} \times \mathbf{b}) \cdot (\nabla \times \boldsymbol{\chi}_\gamma)]\mathrm{d}s. \tag{58}$$

The linear problem (56) determines the harmonic component of the pressure field in incompressible flow problems.

A method more convenient computationally for characterizing the problem (56) is obtained by generalizing the Glowinski–Pironneau method[23] to the present vector context. The fields $\boldsymbol{\chi}_\gamma$ are replaced by the vector functions $\mathbf{w}$ defined by

$$\mathbf{w} = \text{arbitrary in } V, \qquad (\mathbf{n} \cdot \mathbf{w})|_S = \delta(s - \bar{s}), \qquad (\mathbf{n} \times \mathbf{w})|_S = \mathbf{0}, \tag{59}$$

where $\bar{s}$ is contained in S. Then, after problems (54) and (55) have been solved, one solves the vector Helmholtz problems

$$(-\nabla^2 + \gamma)\mathbf{u}_0 = -\nabla p_0 + \gamma \mathbf{u}^n + \mathbf{f}, \qquad \mathbf{u}_0|_S = \mathbf{b} \tag{60}$$

and

$$(-\nabla^2 + \gamma)\mathbf{u}' = -\nabla p', \qquad \mathbf{u}'|_S = \mathbf{0} \tag{61}$$

respectively. Application of the Green identity (50) and integration by parts gives the results

$$A(s, s') = \int [(\nabla \cdot \mathbf{u}')(\nabla \cdot \mathbf{w}) + (\nabla \times \mathbf{u}') \cdot (\nabla \times \mathbf{w}) + \nabla p' \cdot \mathbf{w} + \gamma \mathbf{u}' \cdot \mathbf{w}]\mathrm{d}V, \tag{62}$$

$$\beta(s) = -\int [(\nabla \cdot \mathbf{u}_0)(\nabla \cdot \mathbf{w}) + (\nabla \times \mathbf{u}_0) \cdot (\nabla \times \mathbf{w}) + \nabla p_0 \cdot \mathbf{w} + \gamma \mathbf{u}_0 \cdot \mathbf{w}]\mathrm{d}V + \int (\gamma \mathbf{u}^n + \mathbf{f}) \cdot \mathbf{w}\mathrm{d}V. \tag{63}$$

Unlike the case of the non-primitive variables, the equivalent operator $A(s, s')$ after the spatial discretization is not symmetric. The same happens to the matrix of the influence matrix method for the primitive variables,[30, 32] which is sometimes called the Green function method.[33] It is however possible to make symmetrical the linear problem (56) at the expense of introducing an additional elliptic equation for a scalar potential.[34]

COMPUTATIONAL RESULTS

In this section we shall give some results of calculations carried out to illustrate the methods described in the previous sections. In the first instance two one-dimensional examples are considered and then two illustrations are given of two-dimensional examples treated by the method of series truncation. One of these is taken from an investigation of flow normal to a flat plate of finite breadth by Dennis and Wang Qiang.[35] Finally, an illustration of channel flow computed by Quartapelle and Napolitano[24] is given.

One-dimensional computations

We start with a very simple example which models the biharmonic equation in one space dimension. For this we consider equations (1) and (2) with

$$f(x) \equiv 0, \qquad g(x) = 30x, \qquad l = 1. \tag{64}$$

For such a case we can find a solution $\zeta(x)$ of (1) which satisfies (4) and then determine $\psi(x)$ afterwards by solving (2) subject to any two of the conditions (3). It is easily verified by exact analysis that

$$\zeta(x) = 1 - 9x/2 + 5x^3. \tag{65}$$

Three numerical calculations were carried out, using in each case exactly the same second-order-accurate finite difference formulae (5) and (6) but with different approximations to the boundary values $\zeta(0)$ and $\zeta(l)$. In the first two computations the local conditions (7) and (8) were used and in the third the integral conditions (4a, b) were satisfied approximately by means of quadrature formulae of the type (9) using Simpson's rule.

Some typical computed results are shown in Table I. A grid size $h = 0{\cdot}1$ was used and the three solutions using the analogues (7), (8) and (4a, b) are denoted respectively by A, B and C. Corresponding results calculated from the exact solution are denoted by E.

The approximation C using the integral conditions clearly gives the best results for the vorticity distribution $\zeta(x)$, presumably due to the use of the quadrature formula of superior accuracy in implementing the analogue (9) of the conditions (4a, b). The approximation A, which used only a first-order-accurate local approximation to calculate $\zeta(0)$ and $\zeta(l)$, seems to be uniformly inadequate; but the solution B using a second-order local approximation to the boundary vorticity gives quite an accurate approximation to $\psi(x)$, notwithstanding the fact that the vorticity distribution predicted is not highly accurate. All the computations were carried out using the SOR iterative method for solving (5) and (6) in which one iterative sweep of all grid points using (5) was followed by a similar iterative sweep of (6); then, after the calculation of new boundary conditions for $\zeta(0)$ and $\zeta(l)$, this process was repeated until eventually convergence was achieved. In this process equation (2) was solved as a boundary value problem subject to the conditions $\psi(0) = \psi(1) = 0$.

The approximation D shown in Table I was obtained by retaining the finite difference approximation (5) to equation (1) and treating it as a boundary value problem as before, but equation (2) was solved as a step-by-step problem subject to the conditions

$$\psi(0) = \psi'(0) = 0. \tag{66}$$

Thus, following each SOR iteration of (5), the system of equations

$$\phi' = \zeta, \qquad \phi(0) = 0, \qquad \psi' = \phi, \qquad \psi(0) = 0 \tag{67}$$

was integrated using h^4-accurate formulae. The procedure for obtaining the approximations A, B and C was otherwise kept the same, using conditions (4a, b).

The test that this method is effective is that $\psi(1)$ should come out to be approximately zero. With the same grid size $h = 0{\cdot}1$, the final value of $\psi(1)$ was zero to approximately six decimal places.

Table I. Numerical solutions of equations (1) and (2) for the data of equation (64) using different approximations to the boundary conditions for ζ

	A	B	C	D	E
$\zeta(0{\cdot}0)$	1·0045	1·0044	1·0004	1·0004	1·0000
$\zeta(0{\cdot}5)$	−0·6375	−0·6150	−0·6250	−0·6250	−0·6250
$\zeta(1{\cdot}0)$	1·4704	1·5156	1·4996	1·4996	1·5000
$\psi(0{\cdot}3)$	0·2757	0·2552	0·2649	0·2538	0·2536
$\psi(0{\cdot}5)$	0·4219	0·3937	0·4063	0·3911	0·3906
$\psi(0{\cdot}8)$	0·2039	0·1823	0·1913	0·1799	0·1792

The convergence rate of the iterative sequence was improved enormously and the step-by-step solution of (2) is probably more efficient than a tridiagonal reduction. The solution D for ψ is also more accurate owing to the use of h^4-accurate formulae in the determination of ψ in this case. Of course the use of h^2-accurate approximations to determine ζ followed by h^4-accurate approximations to determine ψ cannot easily be justified in general, but it is not uncommon to use such techniques (see, for example, Loc[36] and Loc and Bouard[37]). The main point in the present work is that the flexibility of using step-by-step methods in the determination of ψ exists when integral conditions are used.

As a second example we consider a non-linear case of equations (1) and (2), namely the Blasius problem for steady flow along a semi-infinite flat plate. The formulation given by Rosenhead[38] is adopted (where the present variable x corresponds to η in Reference 38). The basic equation for $\psi(x)$ is then

$$\psi''' + \psi\psi'' = 0, \tag{68}$$

with

$$\psi(0) = \psi'(0) = 0, \qquad \psi'(\infty) = 1. \tag{69}$$

We can express this problem in the form of the pair of equations (1) and (2) by differentiating (68) once and then expressing the resulting equation as the two equations

$$\zeta'' + \psi\zeta' + \psi'\zeta = 0, \tag{70}$$

$$\psi'' = \zeta. \tag{71}$$

These are comparable with (1) and (2) if we take $f = \psi$, $g = -\psi'\zeta$.

In this problem only the one integral condition of (4) is useful since the second integral is divergent over the infinite domain of the problem. The first condition gives, from the last condition in (69) with the second,

$$\int_0^\infty \zeta \, dx = 1. \tag{72}$$

It is the only condition needed to give an extra equation to determine the boundary value $\zeta(0)$ as part of the solution procedure, since $\zeta \to 0$ as $x \to \infty$. As before, we may express (72) approximately using a quadrature formula of the type (9). Simpson's rule was again used. The infinite limit in (72) must be replaced by a finite but large enough value of x and this is an additional parameter of the problem. The last condition of (69) is also enforced at this finite distance. Thus, finally, equation (71) is solved subject to the boundary conditions $\psi(0) = 0$, $\psi'(\infty) = 1$ and the integral condition is used to determine $\zeta(0)$. The condition $\psi'(0) = 0$ must come out to be satisfied automatically.

Some trial calculations have been carried out using both the h^2-accurate finite difference analogue expressed by equations (5) and (6) and the h^4-accurate analogue given by (10) and (12). Two grid sizes, $h = 0{\cdot}2$ and $h = 0{\cdot}1$, were used and the value $x = 7{\cdot}2$ was taken as the location of application of the conditions at infinity. In the case of the h^2-accurate analogue, all three methods of calculating $\zeta(0)$ using the local approximations (7) and (8) and the integral condition (72) were applied and were found to give essentially the same results. Thus only the results obtained using the integral condition are given in Table II. The solutions A and B in this table give the h^2-accurate results for the grid sizes $h = 0{\cdot}2$ and $0{\cdot}1$ respectively, and the solutions C and D represent the h^4-accurate results. The column E shows the solution given by Rosenhead[38] (p. 224).

The superiority of the h^4-accurate approximation is very clear from this table. The approximation C is more accurate than the approximation B obtained using half the grid size. The fact

Table II. Numerical approximations to the solution of the Blasius problem

	h^2-accurate		h^4-accurate		
	A	B	C	D	E
	$h=0{\cdot}2$	0·1	0·2	0·1	
$\zeta(0)$	0·4639	0·4681	0·4706	0·4698	0·4696
$\psi(1)$	0·2309	0·2324	0·2333	0·2331	0·2330
$\psi(2)$	0·8815	0·8853	0·8877	0·8870	0·8868
$\psi(4)$	2·7767	2·7817	2·7851	2·7842	2·7839
$\psi(6)$	4·7760	4·7810	4·7844	4·7835	4·7832

that the three different methods of calculating $\zeta(0)$ based on equations (7), (8) and (72) give the same approximate solution is not too surprising in this problem, which is a special case. In fact the third and fourth derivatives of ψ at $x=0$ are zero, which makes the approximations (7) and (8) at least of the same order of accuracy and of a higher order of accuracy than they normally would be. The same iterative method of solving the equations that was used in the previous problem, in which one SOR iteration for ζ was followed by one for ψ and then a new value of $\zeta(0)$ calculated, was again adopted here. The alternative method of using a step-by-step method for solving (71) subject to the initial conditions given in (69) was also used. In this problem it gave approximately the same results despite the fact that h^4-accurate step-by-step formulae were used. However, the number of iterations was greatly reduced, e.g. for $h=0{\cdot}2$ only 7% of the iterations needed in the entirely SOR procedure were found to be necessary when a step-by-step integration of (71) was used.

Flow past a circular cylinder and normal to a flat plate

Here we shall give two brief illustrations of results which have been obtained using integral conditions, firstly for flow past a circular cylinder at Reynolds number of 10 and then for flow normal to a flat plate at Reynolds number of 70. The basic method has already been described for both problems in a previous section of the present paper. Two different computations of the flow were carried out in essentially the same manner in each problem. In the first method of computation equations (13) and (14) were solved using a standard h^2-accurate finite difference model with the boundary vorticity at $\xi=0$ calculated using local h^2-accurate approximations following the method of Woods.[3] The only difference in the second method was that the boundary vorticity was calculated using the integral conditions (23) and then computed according to (24). Exactly the same asymptotic expressions for ζ and ψ as $\xi\to\infty$ were used for approximating the boundary conditions at some large enough value of ξ.

In Table III we show comparisons of the surface vorticity at selected locations for the case of flow past a circular cylinder at $Re=10$ obtained on four separate computations. The first two, denoted by A and B, give results obtained using a square grid of side $h=\pi/20$ and the last two, denoted by C and D, were obtained using the square grid $h=\pi/40$. In both cases the boundary $\xi=\xi_\infty$ at which the asymptotic boundary conditions for ζ and ψ were assumed to hold was taken as $\xi=\pi$. The solutions A and C were obtained using the local approximation of Woods[3] to calculate $\zeta(0,\theta)$ and the solutions B and D were calculated using integral conditions.

There is an excellent comparison between the results of the two methods obtained using the grid $h=\pi/40$ and the comparison is satisfactory enough for the results obtained with the coarser grid. The results C and D agree well with those computed by Dennis and Chang.[9] Of course a Reynolds

Table III. Vorticity ($-\zeta(0, \theta)$) on the surface of a circular cylinder for Reynolds number 10

	$h=\pi/20$		$h=\pi/40$	
θ/π	A	B	C	D
0	0	0	0	0
0·1	−0·0516	−0·0610	−0·0497	−0·0495
0·2	0·0452	0·0218	0·0664	0·0653
0·3	0·3680	0·3303	0·4304	0·4271
0·4	0·9093	0·8653	1·0267	1·0218
0·5	1·5671	1·5275	1·7370	1·7320
0·6	2·1330	2·1038	2·3328	2·3286
0·7	2·3492	2·3301	2·5421	2·5389
0·8	2·0323	2·0208	2·1820	2·1797
0·9	1·1818	1·1762	1·2627	1·2614
1·0	0	0	0	0

number of 10 is rather small for this problem, but we have concentrated on the objective of showing that the integral conditions do give satisfactory results. Numerical solutions have in fact been obtained using both methods up to $Re=100$. They will be published later together with details of results obtained using more accurate methods with the object of demonstrating the efficiency of h^4-accurate methods when used in conjunction with integral conditions.

In our second illustration the same two methods are used in the calculation of the flow normal to a flat plate at $Re=70$. These results have been computed by Dennis and Wang Qiang.[35] There are much greater difficulties with the finite difference methods in this problem because of the singularity (mentioned in a previous section) which occurs at $\xi=0$, $\theta=\pi/2$ of the co-ordinate system (25). In fact the finite difference analogue of equation (13) must in some way avoid the infinite vorticity at this singular point. One way is to rotate the operator $\partial^2/\partial\xi^2+\partial^2/\partial\theta^2$ in (13) through half a right angle at the point $\xi=h$, $\theta=\pi/2$, thus avoiding the singularity in deriving finite difference approximations at this special point of a grid formed by lines of constant ξ and θ. This type of technique was used by Dennis and Smith[39] and a satisfactory adaptation of it was applied in the present problem, although we shall not detail it here. At points other than $\xi=0$, $\theta=\pi/2$ the vorticity on the surface of the plate can be calculated using (24) in one method or using the approximation of Woods[3] in the other.

In Table IV we give the surface vorticity at selected locations obtained from the two methods of solution for the case $Re=70$. A quite small grid size $h=\pi/100$ was used in both methods of solution. The solution obtained using purely finite difference methods is denoted by A and that using integral conditions by B. It may be noted that there is a very reasonable comparison between the two solutions except very near to the singularity itself. This is to be expected because of the different methods of dealing with the problem. The important point is that the effect of the singularity is local. However, the use of the integral conditions is a very effective way of dealing with it.

Two-dimensional applications

The numerical applications given in this section have so far been essentially one-dimensional, although some have been derived from two-dimensional problems. There are however numerous

Table IV. Surface vorticity ($-\zeta(0, \theta)$) for flow normal to a flat plate at $Re=70$

θ/π	A	B	θ/π	A	B
0	0	0	0·52	44·712	40·474
0·10	−0·304	−0·308	0·53	26·375	24·317
0·20	−0·675	−0·683	0·55	16·462	16·228
0·30	−1·403	−1·415	0·60	9·784	9·744
0·40	−4·139	−4·126	0·70	5·037	5·017
0·45	−10·422	−10·130	0·80	2·630	2·622
0·47	−19·163	−19·300	0·90	1·143	1·140
0·48	−27·719	−27·168	1·00	0	0

applications which have been made in two dimensions using the formulations of the previous section (see, for example References 20–25). In these various investigations the use of the vorticity integral conditions has allowed the driven cavity problem to be solved by overcoming the difficulties associated with the singularity at the corner using both finite differences and finite elements. Furthermore, the validity of the method for arbitrary domains has been established by means of a finite element spatial discretization using both non-primitive variables[24] and primitive variables.[31].

As an example of a two-dimensional calculation we illustrate the problem of flow in a channel proposed by Roache (see the references in Reference 24). The geometry and boundary conditions are shown in Figure 1, together with the computational grid. The implementation of the vorticity–streamfunction formulation was carried out using a finite element implicit method and the calculations were carried out for a Reynolds number of 10.

The steady state vorticity along the lower wall is shown in Figure 2 and indicates the existence of a separated wall bubble in the expanding region of the channel. The centre of this separated bubble is found to be at $x=1{\cdot}130$, $y=-0{\cdot}6645$ and the corresponding value of the streamfunction there is $\psi=-0{\cdot}0011$. These values are found to be very close to those of a corresponding calculation carried out by a fourth-order accurate spline technique using the ADI method.

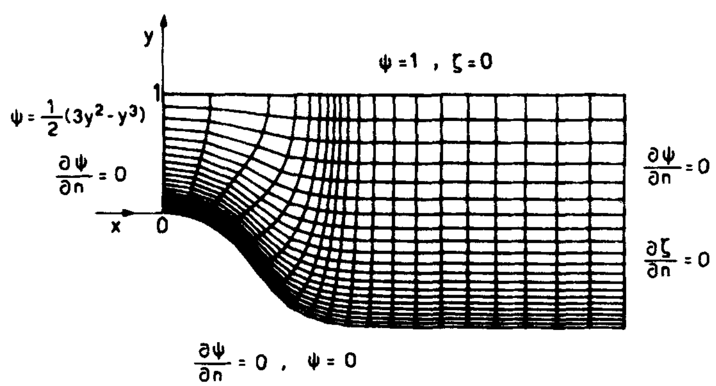

Figure 1. Channel flow geometry, computational grid and boundary conditions

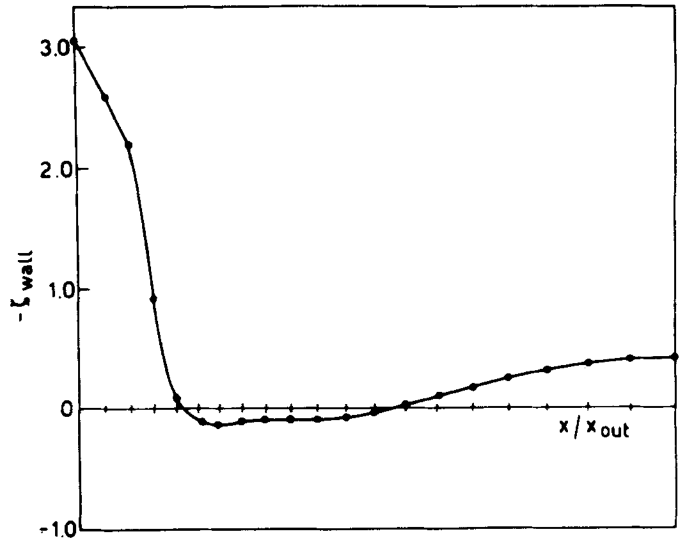

Figure 2. Surface vorticity on the lower wall for channel flow at $Re = 10$

SUMMARY AND CONCLUSIONS

In this paper we have reviewed some methods where, by means of Green's identity and simplified cases equivalent to it, local boundary conditions may be transformed into global conditions, termed integral conditions. Several examples have been considered in one and two space dimensions which indicate that satisfactory results can be obtained using the method. There are of course other examples in the literature in which Green's theorem methods have been used in the numerical solution of Navier–Stokes problems. An early example is the work of Mills[40] and more recently Wang and Wu[41] have used integral conditioning for the vorticity to solve some internal flow problems, including one examined by Mills[40] and Dennis.[42] Mention may also be made of a recent paper by Cerutti *et al.*[43] and the references therein. The most important property of integral conditions seems to be that they apply self-consistent constraints on the vorticity, which gives an effective method of calculating the boundary vorticity from that in the internal flow domain. The same remark applies to the integral conditioning of the pressure when the primitive variable approach is used.

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